Parallel velocity shear instabilities in an inhomogeneous plasma with a sheared magnetic field

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Citation: The Physics of Fluids **16**, 1719 (1973); doi: 10.1063/1.1694200
View online: https://doi.org/10.1063/1.1694200
View Table of Contents: https://aip.scitation.org/toc/pfl/16/10
Published by the American Institute of Physics

Parallel velocity shear instabilities in an inhomogeneous plasma with a sheared magnetic field

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(Received 23 March 1973)

Injection of fast neutrals with a component of the beam velocity along the magnetic field $\mathbf{B}_0$ results in a plasma having a sheared mean velocity along $\mathbf{B}_0$. If this parallel velocity shear U' and the density inhomogeneity scale length L_n are large enough so that $U'|L_n|/(\tau+1)^{1/2}v_i > 1$ then a parallel Kelvin–Helmholtz instability is excited, where v_i is the ion thermal speed and τ is the electron temperature/ion temperature. Magnetic shear stabilization requires a shear length L_s comparable to or less than $|L_n|$, namely, $\tau L_s/3(\tau+1) < v_i/U' < |L_n|/(\tau+1)^{1/2}$. For $U'|L_n|/(\tau+1)^{1/2}v_i < 1$, however, this parallel Kelvin–Helmholtz instability is stabilized and only drift wave instabilities can be excited. For $\tau \approx 1$ and $(M/m)^{1/2} \approx 43$, the first of these drift wave instabilities is unstable for $U'/\Omega_i > 172|(L_n/L_s)^{3/2}|$ provided $|L_n/L_s| < a_i/30|L_n|$, while the second is stable for $U'/\Omega_i < 136a_i/|L_s|$ provided $|L_s/L_n| > 600$ and $U'/\Omega_i \ll 1$, where a_i and Ω_i are the ion gyration radius and frequency.

INTRODUCTION

The injection of fast neutral atoms is presently being investigated as a means of heating toroidal plasmas. Typically, the neutral beam will be injected with a component of its velocity along the magnetic field $\mathbf{B}_0$, and after injection the neutral beam will be ionized, largely by charge exchange. If injection is at or below some critical velocity approximately equal to the electron thermal speed times the cube root of the ratio of the electron upon the ion mass, the energy of these fast beam ions will be collisionally transferred to the plasma ions; otherwise, the beam ions are first slowed down to this critical velocity by the drag of the plasma electrons. In addition to this energy transfer or heating, the momentum of the beam is transferred to the plasma. Consequently, during the time it takes the beam and plasma ions to become indistinguishable, a mean velocity along the magnetic field is setup which differs on adjacent magnetic surfaces because the number of ionizations per second will vary along the path of the neutral beam. This parallel sheared velocity due to the thermalized beam particles requires a nonequilibrium distribution function, and therefore represents a source of free energy which can drive instabilities. The second hump in the distribution function due to beam particles that have not yet been thermalized can also drive instabilities, but these explicitly beam driven instabilities will not be considered. For more information on the neutral beam injection process and on these explicitly beam driven instabilities, Stix[1,2] should be consulted.

Fluid treatments of the parallel sheared velocity driven Kelvin–Helmholtz instability in a plasma supporting a density gradient and immersed in a uniform magnetic field have been previously given by D'Angelo[3] and more recently by Dobrowolny,[4] who also included finite β = particle pressure/magnetic pressure effects. Fluid descriptions, however, cannot properly treat the damping due to the resonant ions which becomes important when the electron temperature T_e becomes comparable to or less than the ion temperature T_i. As a result a kinetic treatment is desirable. Smith and Von Goeler[5] have attempted such a treatment numerically, but have only considered $T_e = T_i$ and did not consider sheared magnetic fields. In this paper a kinetic treatment is presented which analytically includes the effects of the resonant ions, arbitrary T_e/T_i, and magnetic shear, on the parallel Kelvin–Helmholtz instability. In addition, the effects of the parallel velocity shear on the stability of drift waves in a sheared magnetic field is treated. The calculation is performed in a slab geometry in which $\mathbf{B}_0 = B_0(\hat{z} + \hat{y}x/L_s)$ and $U'x$ is the sheared component of the mean velocity along $\mathbf{B}_0$. The density gradient which gives rise to the diamagnetic drift velocities is taken to be in the x direction. The ion and electron U' are taken to be approximately equal so that there is no appreciable net current due to the U'. Furthermore, the Doppler shift due to any constant velocity along $\mathbf{B}_0$ due to the injection is assumed to be negligible.

In the section that follows a simple physical picture of the mechanisms which drive and damp the parallel Kelvin–Helmholtz instability is presented. In the following two sections the equilibrium is discussed and the Vlasov and Poisson equations are solved self-consistently for a low-β plasma in which U' is much less than the ion gyration frequency and the Doppler shift due to U' is small compared with the wave frequency ω of interest. In the subsequent three sections, the local theory ($L_s \to \infty$), the nonlocal corrections to the parallel Kelvin–Helmholtz instability, and the drift wave instabilities are treated.

SIMPLE PHYSICAL PICTURE

In this section a simple physical picture is presented of the mechanisms by which a velocity shear U' can result in instability and a density gradient N' can lead to stabilization. In this simple picture the magnetic

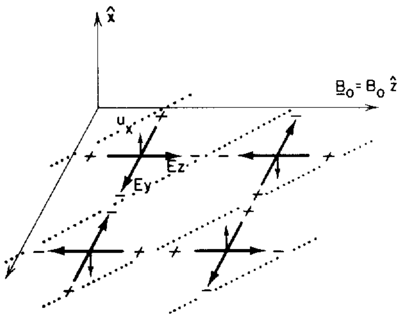

FIG. 1. Sketch of the assumed form of the perturbing electric field $\mathbf{E}=E_y\hat{y}+E_z\hat{z}$.

field $\mathbf{B}_0$ is taken to be constant, the ions are assumed to have a temperature much less than that of the electrons, $T_i \ll T_e$, and the electrons are assumed to be in equilibrium at any time, and at every point. Because of this last assumption, the electrons are Maxwell–Boltzmann so that for constant electron temperature T_e the $\mathbf{E}\times\mathbf{B}_0$ and pressure gradient drifts exactly cancel for the electrons in the presence of any perturbing electrostatic electric field $\mathbf{E}$.

Taking $\mathbf{B}_0$ to be along the $\hat{z}$ axis, the density N and mean velocity along $\mathbf{B}_0$, $\hat{z}\cdot\mathbf{U}$, are assumed to be functions of x only such that $d(\hat{z}\cdot\mathbf{U})/dx\equiv U'$ and $dN/dx\equiv N'$ are constant over the x distances of interest. In addition, $U'>0$ and $N'>0$ may be assumed. If a perturbing field $\mathbf{E}=-i(k_y\hat{y}+k_z\hat{z})\phi$ having no x variation is assumed as shown in Fig. 1, then the ions will try to move along $\mathbf{B}_0$ to neutralize the charge. As a result, they will undergo a change in velocity $\delta u_z^{(1)}=-(e/M)ik_z\phi\delta t$ in a time δt, where e and M are the charge and mass of the ions. Because of the $\mathbf{E}\times\mathbf{B}_0$ drift of the ions, $u_x=-ik_y\phi c/B_0$, and the velocity shear U'; however, at any given x the ions whose velocities are increased (decreased) by $\hat{z}\cdot\mathbf{E}$ are replaced by slower (faster) ions from smaller (larger) x, provided $k_z/k_y>0$ as shown in Fig. 1. Therefore, during the time δt the ions also undergo an opposing change in velocity due to U', $\delta u_z^{(2)}=-(u_x\delta t)U'$. Consequently, the parallel Kelvin–Helmholtz instability occurs when $|\delta u_z^{(2)}|>|\delta u_z^{(1)}|$, that is, when $U'/\Omega_i>k_z/k_y$, where Ω_i is the ion gyration frequency. Note that the $k_z/k_y<0$ case is always stable as $\delta u^{(2)}$ is in the same direction as $\delta u_z^{(1)}$. The preceding picture for the parallel Kelvin–Helmholtz instability is essentially the same as that first given by Rome and Briggs.[6]

In order to understand the role of the density gradient in stabilizing the parallel Kelvin–Helmholtz instability it is convenient to consider the limit $U'/\Omega_i\gg k_z/k_y>0$ in which the $\hat{z}\cdot\mathbf{E}$ stabilizing force is negligible. In this limit the growth rate γ of the Kelvin–Helmholtz instability can be determined by noting that the divergence in the ion velocity results in a density change $\delta n^{(2)}=-Nik_zu_z^{(2)}\delta t$, where $u_z^{(2)}=-\int dtu_xU'\approx -u_xU'/\gamma$. Consequently, $n^{(2)}=-\int Nik_zu_z^{(2)}\,dt\approx -Nik_zu_z^{(2)}/\gamma$, and from quasineutrality $n^{(2)}$ must equal the perturbed electron density $Ne\phi/KT_e$. As a result, the growth rate γ is given by $\gamma^2\approx k_zk_y(KT_e/M)(U'/\Omega_i)$, where K is Boltzmann's constant.

In estimating γ, the density perturbation caused by N' has been neglected. Because of N', however, the ions which $\mathbf{E}\times\mathbf{B}_0$ drift to larger x are replaced by fewer ions, and those that drift to smaller x are replaced by more. The resulting perturbation in the ion density in a time δt is just $\delta n^{(3)}=-(u_x\delta t)N'$. Setting $n^{(3)}=-\int dtu_xN'\approx -u_xN'/\omega$ equal to the perturbed electron density $Ne\phi/KT_e$ gives $|\omega|\approx k_yKT_eN'/NM\Omega_i\equiv|\omega_*|$, where ω_* is the electron diamagnetic drift frequency. Consequently, the time scale associated with the changes in density due to N' is just the time required for the electrons, and therefore $\mathbf{E}$, to "drift" a wavelength $2\pi k_y^{-1}$. Referring to Fig. 1, this means that in a time $2\pi|\omega_*|^{-1}$ the pattern drifts a full wavelength in the minus $\hat{y}$ direction. As a result, if $\mathbf{E}$ drifts a distance πk_y^{-1} in a time $\pi|\omega_*|^{-1}$ short compared with the growth time γ^{-1}, then the parallel Kelvin–Helmholtz instability will not occur because the ions moving along $\mathbf{B}_0$ can no longer pile up appreciable amounts of positive charge before the diamagnetic drift of the electrons will result in it being neutralized. For stability therefore, $\gamma^2\ll\omega_*^2$ or $U'/\Omega_i<(k_yKT_e/k_zM\Omega_i^2)(N'/N)^2$ is required. Note that the stabilization is independent of the sign of N'. For $N'<0$, the electron diamagnetic drift is in the plus $\hat{y}$ direction.

Although $T_e\gg T_i$ has been assumed in the preceding discussion, the mechanisms described are expected to be qualitatively correct when the ion diamagnetic drift becomes comparable to or greater than that of the electrons and when resonant ion effects can no longer be neglected, that is, for $T_e\lesssim T_i$.

EQUILIBRIUM

For a collisionless, inhomogeneous, low-β plasma having a sheared mean particle velocity $\mathbf{U}(x)$ along the sheared magnetic field $\mathbf{B}_0=B_0(\hat{z}+\hat{y}x/L_s)$ the particle trajectories must satisfy the equations of motion and initial conditions

$$\frac{d\mathbf{r}'}{dt'}=\mathbf{v}',\qquad \mathbf{r}'(t'=t)=\mathbf{r},$$
$$\frac{d\mathbf{v}'}{dt'}=\frac{e}{Mc}\mathbf{v}'\times\mathbf{B}_0(x'),\qquad \mathbf{v}'(t'=t)=\mathbf{v}, \tag{1}$$

where primes are used to denote trajectory variables. The constants c, e, and M are the speed of light, charge, and mass; and species subscripts are suppressed. Equations (1) yield conservation of energy and two components of canonical momentum;

$$|\mathbf{v}'|^2=|\mathbf{v}|^2=v^2,\qquad x'+(v_y'/\Omega)=x+(v_y/\Omega),$$
$$v_z'+\frac{x'}{L_s}v_y'+\frac{v_y'^2}{2\Omega L_s}=v_z+\frac{x}{L_s}v_y+\frac{v_y^2}{2\Omega L_s}, \tag{2}$$

where $\Omega = eB_0/Mc$ and L_s is the magnetic shear length. As a function of the constants of the motion (2), the equilibrium distribution function f_0 having the appropriate sheared mean particle velocity along $\mathbf{B}_0$ and the desired density inhomogenity is

$$f_0 = \frac{N_0}{(2\pi v_T^2)^{3/2}} \exp\left\{\frac{[x+(v_y/\Omega)]}{L_n} - \left[v^2 - 2\left(v_z + \frac{x}{L_s}v_y + \frac{v_y^2}{2\Omega L_s}\right)\left[u+U'\left(x+\frac{v_y}{\Omega}\right)\right] + \left[u+U'\left(x+\frac{v_y}{\Omega}\right)\right]^2\right](2v_T^2)^{-1}\right\}.$$

The $\hat{z}$ and $\hat{x}$ axes are chosen such that $B_0 > 0$ and $U' > 0$, respectively. The plane $x=0$ corresponds to a rational surface and is located at the point at which the wave vector $\mathbf{k}$ under consideration has no component along $\mathbf{B}_0$. The quantities L_n, $v_T = (KT/M)^{1/2}$, and N_0 are the scale length of the density inhomogenity, the thermal speed of particles of mass M at temperature T, and the particle density at $x=0$.

The current that results in the magnetic shear is assumed to be due to the electron u along. Consequently, $u=0$ for the ions. The ion and electron U' are assumed to be approximately equal so that currents due to the sheared part of the particle mean velocities, as well as diamagnetic currents, are assumed small compared with $|eNu|$.

The desired first two moments of f_0,

$$N = N(x) = N_0 \exp(x/L_n)$$

and (3)

$$\mathbf{U} = \mathbf{U}(x) = (v_T^2/\Omega L_n)\hat{y} + (u+U'x)[\hat{z} + (x/L_s)\hat{y}],$$

are obtained provided

$$U'/\Omega_i \ll 1, \quad |x/L_s| \ll 1, \quad a_i/|L_s| \ll 1, \quad |u|/v_e \lll 1. \tag{4}$$

In Eq. (4), $v_e = (KT_e/m)^{1/2}$ and $v_i = (KT_i/M)^{1/2}$, are the electron and ion thermal speeds, and $a_i = v_i/\Omega_i$, is the ion gyroradius with $\Omega_i = eB_0/Mc$. In addition to giving (3), f_0 is constructed so that neglecting gyroradius corrections, $a_i/|x| \ll 1$, and using (4) results in the drifting Maxwellian

$$f_0 = \frac{N(x)}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{v_x^2 + v_y^2 + [v_z - (u+U'x)]^2}{2v_T^2}\right). \tag{5}$$

LINEARIZED EQUATIONS

In terms of the trajectory variables of (1), the solution of the Vlasov equation for the perturbed distribution function f can be cast into the form

$$f = f(\mathbf{r}, \mathbf{v}, t) = \frac{e}{M}\int_{-\infty}^{0} d\tau\, \nabla'\Phi(\mathbf{r}', \tau+t=t') \cdot \nabla_{v'} f_0(x', \mathbf{v}').$$

Using the constants of the motion (2) and $\mathbf{v}' \cdot \nabla'\Phi(\mathbf{r}', \tau+t) = d\Phi/d\tau - \partial\Phi/\partial\tau$, neglecting gyroradius corrections and using inequalities (4), and seeking solutions of the form $\exp(iky - i\omega t)$ gives

$$f = f(x, k, \mathbf{v}, \omega) = -(e/KT)f_0\{\Phi(x, k, \omega) + i[\omega - \omega_d - k_{\|}(u+U'x) - (kU'/\Omega)(v_z - u - U'x)]I(x, k, \omega)\}, \tag{6}$$

where the $x=0$ plane is located such that $\mathbf{k}\cdot\mathbf{B}_0(x=0) = 0$, and the parallel wave vector $k_{\|}$ is defined as $k_{\|} = kx/L_s$. In Eq. (6)

$$I = I(x, k, \omega) = \int_{-\infty}^{0} d\tau \Phi(x', k, \omega) \times \exp[-i\omega\tau + ik(y'-y)], \tag{7}$$

$\omega_d = kv_T^2/\Omega L_n$ is the particle diamagnetic drift frequency, and f_0 as given by (5) may be employed. Note that there is a component of the wave vector $k\hat{y}$ along $\mathbf{B}_0$ for all x except $x=0$ so that the parallel Kelvin–Helmholtz instability may be excited.

For potentials Φ that vary slowly compared with the particle gyroradius, $\Phi(x', k, \omega) = \Phi(x')$ may be Taylor expanded about x. In addition, the solutions of (1),

$$x' - x = (v_\perp/\Omega)[\sin(\Omega\tau - \varphi) + \sin\varphi],$$

$$y' - y = (v_\perp/\Omega)[\cos(\Omega\tau - \varphi) - \cos\varphi] + (xv_z\tau/L_s),$$

$$v_x = v_\perp \cos\varphi, \qquad v_y = v_\perp \sin\varphi,$$

the generating function

$$\exp(i\eta\cos\alpha) = \sum J_l(\eta) \exp[il\alpha + il(\pi/2)],$$

and the recurrence relations for the Bessel functions $J_l(\eta)$ may be employed in carrying out the τ integration to obtain

$$I = \sum_{p=-\infty}^{\infty} (-i)^p J_p\left(\frac{kv_\perp}{\Omega}\right)\exp(ip\varphi) \times \sum_{l=-\infty}^{\infty} \frac{i^l \exp(-il\varphi)}{-i(\omega - k_{\|}v_z - l\Omega)} \left\{J_l\left(\frac{kv_\perp}{\Omega}\right) \times \left[\Phi + \left(\frac{v_\perp}{\Omega}\sin\varphi - \frac{l}{k}\right)\frac{\partial\Phi}{\partial x} + \frac{1}{2}\left(\frac{l^2}{k^2} - \frac{2lv_\perp}{k\Omega}\sin\varphi + \frac{v_\perp^2}{\Omega^2}\sin^2\varphi\right)\frac{\partial^2\Phi}{\partial x^2}\right] - \frac{v_\perp}{2k^2}\left[\frac{\partial}{\partial v_\perp}J_l\left(\frac{kv_\perp}{\Omega}\right)\right]\frac{\partial^2\Phi}{\partial x^2}\right\}. \tag{8}$$

In order to obtain the differential equation for Φ, $\int d^3v\, f = \int d\varphi\, dv_\perp\, dv_z v_\perp f$ must be evaluated. Making the additional assumptions $|\omega|/\Omega_i$ and $k^2a_i^2 \ll 1$ the φ integration is performed first. The integrations over $v_\perp$ and finally, v_z then result in

$$e\int d^3v f = -\frac{e^2N(x)}{KT}\times\left(\Phi+\left\{\frac{\tilde{\omega}-\omega_d}{\tilde{\omega}}\eta Z(\eta)-\frac{kU'}{k_{\|}\Omega}[1+\eta Z(\eta)]\right\}\times\left[\left(1-\frac{k^2v_T^2}{\Omega^2}\right)\Phi+\frac{v_T^2}{\Omega^2}\frac{\partial^2\Phi}{\partial x^2}\right]\right), \quad (9)$$

where $\tilde{\omega}=\omega-k_{\|}(u+U'x)$ and $\eta=\tilde{\omega}/|k_{\|}|v_T$. In Eq. (9), $Z(\eta)$ is the plasma dispersion function which is defined here as

$$Z(\eta)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\xi\frac{\exp(-\frac{1}{2}\xi^2)}{\xi-\eta}$$

for $\mathrm{Im}\eta>0$, and must be analytically continued to $\mathrm{Im}\eta\leq 0$.[7]

For a two-component electron-ion plasma composed of singly charged ions, substituting $e\int d^3vf$ into Poisson's equation results in the differential equation

$$a_i^2\frac{\partial^2\Phi}{\partial x^2}+\left((1-k^2a_i^2)+\frac{(\tau+1)+\{[\omega-\omega_*-k_{\|}(u+U'x)]/|k_{\|}|v_e\}Z(\omega/|k_{\|}|v_e)}{(\tau+\omega_*/\omega)\zeta Z(\zeta)-(\tau kU'/k_{\|}\Omega_i)[1+\zeta Z(\zeta)]}\right)\Phi=0, \quad (10)$$

where $\zeta=\omega/|k_{\|}|v_i$, provided the Debye length is small compared with the ion gyroradius. In obtaining Eq. (10) electron terms smaller than the corresponding ion terms by the ratio of the electron mass m to the ion mass M are neglected, the Doppler shifts due to U' are assumed small enough so that

$$|k_{\|}U'x|\ll|\omega| \quad\text{and}\quad |\omega_*|/\tau, \quad (11)$$

and the Doppler shift due to u is neglected compared with ω. In writing Eq. (10) the definitions

$$\tau=T_e/T_i \quad\text{and}\quad \omega_*=-kv_e^2/\Omega_eL_n=-\tau ka_iv_i/L_n \quad (12)$$

are employed, where $\Omega_e=eB_0/mc$.

In order to proceed with the analysis conveniently, it is desirable to have an approximation for $Z(\zeta)$ which is valid in the drift limit $k_{\|}^2v_i^2\ll|\omega^2|\ll k_{\|}^2v_e^2$ and which also approximates $Z(\zeta)$ fairly well for $|\zeta|=|\omega/k_{\|}v_i|\lesssim 1$. Letting $\zeta=i\xi$ for convenience, such an approximation can be formed by the ratio of two polynomials

$$\{[(\tfrac{1}{2}\pi)^{1/2}+\xi]/[1+(\tfrac{1}{2}\pi)^{1/2}\xi+\xi^2]\} \quad (13)$$

provided $\mathrm{Im}\zeta>0$ ($\mathrm{Im}\omega>0$), and $|\mathrm{Re}\zeta|\gg|\mathrm{Im}\zeta|$ ($|\mathrm{Re}\omega|\gg|\mathrm{Im}\omega|$) when $|\zeta|^2=|\omega/k_{\|}v_i|^2\gg 1$. In addition to reproducing the first two terms of the asymptotic expansion of Z for large argument, the approximation (13) gives correctly the first term of the power series expansion of Z for small argument.

Substituting (13) into (10), defining $\omega\equiv i\gamma$, and neglecting resonant electron effects to lowest significant order in the small parameter $|[\omega-\omega_*-k_{\|}(u+U'x)]/k_{\|}v_e|$, results in

$$a_i^2\frac{\partial^2\Phi}{\partial x^2}-V(x,\gamma)\Phi=0, \quad (14)$$

where

$$V=V(x,\gamma)=(L\pm Px+Ax^2)/(T\pm Rx) \quad (15)$$

with

$$T=\tau+(\omega_*/i\gamma), \qquad R=(kv_i/\gamma L_s)[T(\tfrac{1}{2}\pi)^{1/2}\pm W],$$

$$A=(\tau+1)(kv_i/\gamma L_s)^2, \qquad L=1-(\omega_*/i\gamma)+Tk^2a_i^2,$$

$$P=(kv_i/\gamma L_s)[L(\tfrac{1}{2}\pi)^{1/2}\mp W(1-k^2a_i^2)],$$

and

$$W=\tau ka_iU'/\gamma.$$

In the preceding, k may be taken to be positive for unstable solutions ($\mathrm{Im}\omega>0$) as can be seen by letting $k\to -k$ and $\omega\to-\omega^*$ in the complex conjugate of (10). In addition, L_s may be taken to be positive when resonant electron effects are negligible as $L_s\to -L_s$ leaves Eq. (10) unchanged provided $x\to -x$. As a result, $|k_{\|}|=(k/L_s)|x|=\pm kx/L_s$ is employed so that the upper sign in the preceding expressions is for $x>0$ while the lower is for $x<0$.

STABILITY ANALYSIS: LOCAL THEORY

In the limit of large L_s, the local approximation to the differential equation (14) may be found. This local dispersion equation is recovered by solving $\partial V/\partial x=0$ for the extremal point x_0. Then, $V(x,\gamma)$ is expanded about x_0 and terms quadratic in $(x-x_0)$ are retained. The resulting differential equation for Weber functions has a solution going to zero for large $|x-x_0|$ provided an eigenvalue equation is satisfied. As will be shown in subsequent sections, this eigenvalue equation reduces to the local dispersion equation $V(x_0,\gamma)=0$ for large enough L_s.

In order to proceed analytically only the two limits $|W/T|\ll 1$ and $|W/T|\gg 1$ will be considered. In the $|W/T|\ll 1$ limit $x_0=(\gamma L_s/kv_i)[W/2T-O(W^2/8T^2)]$ and the local dispersion equation $V(x_0,\gamma)=0$ may be written as

$$\left[1-\frac{\omega_*}{\omega}+\left(\tau+\frac{\omega_*}{\omega}\right)k^2a_i^2\right]+\frac{W^2}{4T^2}\left[1-\tau-\frac{2\omega_*}{\omega}+2\left(\tau+\frac{\omega_*}{\omega}\right)k^2a_i^2\right]=0. \quad (16)$$

Because $|W/T|\ll 1$, the two terms in square brackets in (16) can balance only if the first bracket is much smaller than the second in magnitude. Recalling that $k^2a_i^2\ll 1$, the only two possibilities that exist are $\omega\approx\omega_*$ and $\tau\gg 1$, $|\omega_*/\omega|$. Considering this $\tau\gg 1$ limit first and using $k^2a_i^2 \ll 1$ gives

$$\omega=\omega_* \frac{1\pm[1-(U'^2L_n^2/\tau v_i^2)(1+\tau k^2a_i^2)]^{1/2}}{2[1+\tau k^2a_i^2]}. \quad (17)$$

Consequently, stability requires

$$U'/\Omega_i<(a_i/|L_n|)[\tau/(1+\tau k^2a_i^2)]^{1/2}.$$

As a result, even in the presence of a sheared $\mathbf{B}_0$ a density gradient tends to stabilize the parallel Kelvin–Helmholtz instability when $\tau \gg 1$ and $|\omega_*/\omega|$. When $(U'^2L_n^2/\tau v_i^2)\gg 1$, Eq. (17) reduces to

$$\gamma=\tfrac{1}{2}\tau^{1/2}kU'a_i=|\omega_* U'L_n/2\tau^{1/2}v_i|\gg\tfrac{1}{2}|\omega_*| \quad (18)$$

so that the parallel Kelvin–Helmholtz instability is quite strong compared with drift wave instabilities. In this large U' limit, inequalities (11) impose on L_s the restriction $(U'/\Omega_i)^2 \ll 2\tau^{1/2}a_i/L_s$.

As mentioned previously, the only other limit in which a solution to (16) exists is $\omega \approx \omega_*$. Taking $\omega=\omega_*(1+\epsilon)$ with $|\epsilon|\ll 1$ gives the dispersion relation for drift waves corrected to include the effect of the velocity shear,

$$\omega=\omega_*\{1-(\tau+1)k^2a_i^2-[U'^2L_n^2/4(\tau+1)v_i^2]\}. \quad (19)$$

The interesting feature of (19) is that the term in U' acts to lower ω from ω_* and so might be expected to result in an instability in a manner similar to that of the $k^2a_i^2$. This drift mode will be considered in more detail in a later section.

For $|W/T|\gg[2+(\pi/2)^{1/2}]$, $x_0=(\gamma L_s/kv_i)\{\pm 1-(T/W)[1+\frac{1}{2}(\pi/2)^{1/2}]+O(T^2/W^2)\}$ and the local dispersion equation $V(x_0,\gamma)=0$ reduces to

$$\pm W(1-k^2a_i^2)=[2+\tfrac{1}{2}\pi)^{1/2}](\tau+1). \quad (20)$$

As a result,

$$\gamma=\pm\tau kU'a_i(1-k^2a_i^2)/[2+(\tfrac{1}{2}\pi)^{1/2}](\tau+1)$$
$$\approx\pm\tau kU'a_i/3(\tau+1)=\mp\omega_* U'L_n/3(\tau+1)v_i. \quad (21)$$

In this $|W/T|\gg 1$ limit, therefore, the parallel Kelvin–Helmholtz instability persists for the $x_0>0$ extremum, but it appears that it can no longer be stabilized by simply decreasing $|L_n|$ or increasing τ. However, using (20) in $|W/T|\gg[2+(\pi/2)^{1/2}]\approx 3$ gives

$$[(\tau/\tau+1)^2+(3v_i/U'L_n)^2]^{1/2}\ll 1.$$

Consequently, the dispersion equation (20), and therefore the root given by (21), is valid only if $\tau \lesssim 1$ and $U'|L_n|\gg 3v_i$. By using these two inequalities the instability (21) is again found to have a growth rate large compared with $|\omega_*|$, $|\gamma|\gg|\omega_*|/(\tau+1)$. Furthermore, because of these two inequalities, it is not surprising that L_n can only result in a higher order correction to γ in this $|W/T|\gg 1$ limit. In addition, note that inequalities (11) impose on L_s the restriction $(U'/\Omega_i)^2 \ll 3(\tau+1)a_i/\tau L_s$.

Consistent with the preceding $|W/T|\gg 1$ limit and the two cases of the $|W/T|\ll 1$ limit, the parallel Kelvin–Helmholtz instability of (17) or (18) and (21) may be stabilized by decreasing $|L_n|$ until

$$U'|L_n|/(\tau+1)^{1/2}v_i<1, \quad (22)$$

for arbitrary τ. When (22) is satisfied, the parallel Kelvin–Helmholtz instability goes over into a drift wave (19).

By replacing $k_\parallel$ with k_z the local theory in the absence of magnetic shear can be examined. Analyzing this $\mathbf{B}_0=\text{const}$ case for $k_z/k>0$ results in the stability condition

$$\frac{U'}{\Omega_i}<\frac{k_z}{k}\left[\frac{\tau+1}{\tau}+\frac{\tau k^2a_i^2}{4k_z^2L_n^2}\right],$$

consistent with the limits analogous to those treated in the preceding. The $k_z/k<0$ case is always stable. The preceding is in agreement with the two-fluid results of D'Angelo[3] and Dobrowolny[4] in the $\tau \gg 1$ limit, and with the results of Smith and von Goeler[5] for $\tau \equiv 1$. Within constants, it also agrees with the results of the simple physical picture presented in an earlier section. It should be pointed out that a two-fluid treatment is equivalent to employing the equations of magnetohydrodynamic with $c\mathbf{E}+\mathbf{U}\times\mathbf{B}_0=-c\nabla p/eN+\mathbf{J}\times\mathbf{B}/eN$ rather than $c\mathbf{E}+\mathbf{U}\times\mathbf{B}_0=0$, where p, $\mathbf{J}$, N, and $\mathbf{U}$ are the pressure of the electrons, the current density, the particle density, and the mean velocity of the ions, respectively.

STABILITY ANALYSIS: NONLOCAL THEORY

So far only the local approximation to the differential equation (14) has been considered and in this small magnetic shear limit a density gradient is found to be stabilizing. It is, however, of interest to determine if the nonlocal behavior caused by larger amounts of magnetic shear can stabilize the purely growing parallel velocity shear driven Kelvin–Helmholtz instability in the absence of a significant density gradient.

For $|L_n|\to\infty$, $k^2a_i^2\to 0$, and γ pure real and >0, an examination of (10) or (15) shows that $V(x,\gamma)$ either has or does not have a pole at some $x<0$, depending upon whether W/τ is greater or less than a number approximately equal to one, respectively.

When $|W/\tau|\ll 2$, $V(x,\gamma)$ can be expanded about x_0. Retaining up to terms quadratic in $(x-x_0)$, Eq. (14) becomes the differential equation for the Weber functions D_n,

$$\frac{\partial^2\Phi}{\partial x^2}-[\lambda+\mu^2(x-x_0)^2]\Phi=0, \quad (23)$$

where

$$\lambda=V(x_0,\gamma)/a_i^2=a_i^{-2}(\tau^{-1}-W^2/4\tau^2)$$

and

$$\mu^2=\frac{1}{2a_i^2}\frac{\partial^2V(x,\gamma)}{\partial x^2}\bigg|_{x=x_0}=(k\Omega_i/\gamma L_s)^2.$$

Because $|W/\tau|\ll 2$, for a local well about x_0, that is, for $V(x_0,\gamma)<0$, $\tau \gg 1$ is required. The solution satisfying (23) and going to zero for large $|x|$ at any fixed

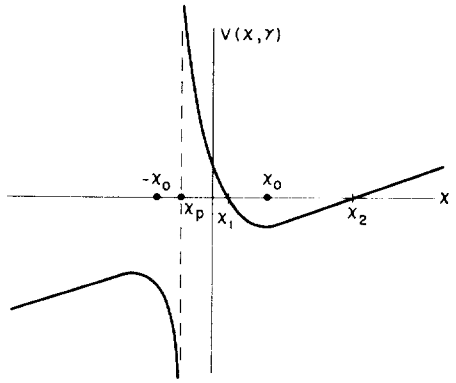

FIG. 2. Plot of $V(x, \gamma)$ vs x for $|W| \gg \tau+1$ and γ real and >0, in the $|L_n| \to \infty$ and $k^2a_i^2 \to 0$ limit.

time t is

$$\Phi = H_n\{[(\pm)\mu(x-x_0)^2]^{1/2}\} \exp[(\mp)\tfrac{1}{2}\mu(x-x_0)^2] = 2^{n/2}D_n\{[(\pm)2\mu(x-x_0)^2]^{1/2}\} \quad (24)$$

provided the eigenvalue equation

$$\lambda(\pm)(2n+1)\mu = 0 \quad (25)$$

is satisfied. In Eqs. (24) and (25) the upper and lower signs in the parenthesis are to be used when $\mathrm{Re}\mu > 0$ and $\mathrm{Re}\mu < 0$, respectively; and H_n is the nth Hermite polynomial. For $\mathrm{Re}\gamma = \mathrm{Im}\omega > 0$, $\mathrm{Re}\mu > 0$ so the upper signs must be used and (25) gives

$$\gamma = -(2n+1)(\tau k a_i v_i/2L_s) + \{[(2n+1)(\tau k a_i v_i/2L_s)]^2 + \tfrac{1}{4}\tau k^2 a_i^2 U'^2\}^{1/2}. \quad (26)$$

When $2(U'L_s)^2 \gg (2n+1)^2 \tau v_i^2$, (26) reduces to the local result (18). For substantial stabilization $(U'L_s)^2 \ll 2(2n+1)^2 \tau v_i^2$ is required, in which case (26) reduces to

$$\gamma = (\tfrac{1}{2}\tau^{1/2} k U' a_i)[U'L_s/2(2n+1)\tau^{1/2} v_i]. \quad (27)$$

From $(W/2\tau)^2 \ll 1$ and $(U'L_s)^2 \ll 2(2n+1)^2 \tau v_i^2$, however, (27) holds only for L_s such that

$$1 \ll [U'L_s/2(2n+1)v_i]^2 \ll \tau/4.$$

As a result, only for very large τ can the $\tau \gg 1$ root (18) be substantially stabilized by magnetic shear in this $|W/\tau| \ll 2$ limit. For smaller values of $U'L_s/v_i$, the inequality is reversed so that the $|W/\tau| \gg 1$ limit must be considered.

Because of the pole at some $x<0$, $V(x, \gamma)$ in the $|W/\tau| \gg 1$ limit has a form somewhat more complicated than the simple well of the $W/\tau \ll 1$ limit. For γ pure real and >0, the form of $V(x, \gamma)$ is as shown in Fig. 2, where the plot is obtained by the following considerations. In order to have a local well for $x>0$, that is, in order for $V(x_0 \approx \gamma L_s/kv_i, \gamma) < 0$, $|W| > (\pi/2)^{1/2}$ is required. Consequently, using $|W| \gg \tau+1$ in Eq. (15) results in

$$V = \left[1 - \frac{Wkv_i x}{\gamma L_s} + (\tau+1)\left(\frac{kv_i x}{\gamma L_s}\right)^2\right]\left(\tau + \frac{Wkv_i x}{\gamma L_s}\right)^{-1}, \quad (28)$$

which has the two simpler limits

$$V = [(\tau+1)kv_i x/W\gamma L_s] - 1 = (x/x_2) - 1 \qquad \text{for } |x| > |\gamma L_s/kv_i| \quad (29)$$

and

$$V = \frac{1-(Wkv_i x/\gamma L_s)}{\tau + (Wkv_i x/\gamma L_s)} = \frac{1-(x/x_1)}{\tau(1-x/x_p)} \qquad \text{for } |x| < \left|\frac{\gamma L_s}{kv_i}\right|. \quad (30)$$

From the form of V as shown in Fig. 2, a physically acceptable solution for $\mathrm{Re}\gamma > 0$ is one which decays for $x > x_2$, oscillates between x_1 and x_2, and leaks through the "barrier" between x_1 and x_p to give an oscillatory solution for $x < x_p$ which goes to zero as $x \to -\infty$. Subject to such behavior at infinity, the differential equation (14) can be solved in the $x > |\gamma L_s/kv_i|$ and $x < |\gamma L_s/kv_i|$ limits in which (29) and (30) are applicable. An eigenvalue equation can then be found by using the WKB solution in the region about $x \approx |\gamma L_s/kv_i|$ to match these solutions. The complete derivation of the eigenvalue equation is presented in the appendix. Also contained in the appendix is an analysis of the eigenvalue equation which shows that the amount of magnetic shear necessary for stabilization is given approximately by

$$\tau U'L_s/(\tau+1)v_i < \tfrac{9}{8}\pi, \quad (31)$$

independent of any restriction on the value of τ compared to one. As a result, magnetic shear stabilizes the parallel Kelvin-Helmholtz instability when (31) is satisfied, although for the case of very large τ, stabilization via (26) first occurs. Note that (31) is only valid if (11) is satisfied with $|x| \lesssim |x_2|$, that is, if $1 \gg |k_{\|}(x_2)U'x_2/\gamma| \approx (3U'/\Omega_i)[\tau U'L_s/(\tau+1)v_i]$.

In order for the local solution to be valid, the barrier width $|x_1 - x_p|$ of Fig. 2 must be large compared with the decay length $|(2\mu)^{-1/2}|$ implied by (23) and (24). For the local solution (21), this gives approximately $\tau U'L_s \gg 8(\tau+1)v_i$, in agreement with the result found in the appendix.

STABILITY ANALYSIS: DRIFT LIMIT

If L_n is small enough, that is, if $U'|L_n|/(\tau+1)^{1/2}v_i < 1$, then the parallel Kelvin-Helmholtz instability of (17), (18), or (21) is stabilized by the density gradient. In this limit, however, drift waves, (19), can be unstable. The balancing in Eq. (10) of the nonlocal behavior introduced by the magnetic shear and the drift wave resonance with the electrons, as represented by the imaginary part of the electron Z function, determine the stability of the drift mode.

To determine the nonlocal corrections, $V(x, \gamma)$ is expanded about $x_0 = (\gamma L_s/kv_i)[W/2T - O(W^2/8T^2)]$ and terms up to these quadratic in $(x-x_0)$ are retained. Then, $V(x_0, \gamma)$ is expanded about the frequency

$\gamma_0=-i\omega_0$ given by (19) and satisfying $V(x_0,\gamma_0)=0$. Only the leading non-zero term is retained. The result of these operations is again of the form of (23) but with

$$\mu^2=\frac{1}{2a_i^2}\frac{\partial^2V(x,\gamma)}{\partial x^2}\bigg|_{x_0,\gamma_0}=-\left(\frac{\tau+1}{\tau+(\omega_*/\omega)}\right)\left(\frac{k\Omega_i}{\omega L_s}\right)^2\bigg|_{\omega\approx\omega_*}$$
$$=-\left(\frac{L_n}{a_i^2\tau L_s}\right)^2 \quad (32)$$

and

$$\lambda=\frac{\delta\gamma}{a_i^2}\frac{\partial V(x_0,\gamma)}{\partial\gamma}\bigg|_{\gamma_0}=\frac{i\delta\gamma}{a_i^2(\tau+1)\omega_*}.$$

In order for the solution (24) of Eq. (23) to go to zero as $|x|\to\infty$ for an unstable eigenvalue ($\mathrm{Im}\omega>0$) and for the λ and μ^2 of (32), the eigenvalue equation (25) must be satisfied with $(\pm)\mathrm{Re}\mu(\omega\approx\omega_*+i\ \mathrm{Im}\omega)>0$. Consequently, the upper or lower signs in the parenthesis of (24) and (25) must be employed depending upon whether $L_s>0$ or <0, with the result that (25) gives

$$\delta\gamma=-(2n+1)[(\tau+1)/\tau]\,|L_n\omega_*/L_s|, \quad (33)$$

where both signs of L_s must be considered because the electron Z function is to be retained. Note that if $V(x_0,\gamma)$ had not been expanded about γ_0, then $\lambda=V(x_0,\gamma)/a_i^2$, and from (10) the terms in $V(x_0,\gamma)$ are of order one. As a result, (25) then gives the local result $V(x_0,\gamma_0)=0$ as being valid when $a_i^{-2}\gg(2n+1)\,|\mu|$ or $\tau L_s\gg(2n+1)\,|L_n|$.

The lowest significant order contribution from the electron Z function that appears in (10) is its residue. Retention of this residue in the drift limit results in a differential equation of the form

$$\frac{\partial^2\hat{\Phi}}{\partial x^2}-[\hat{\lambda}+\mu^2(x-x_0)^2+iY(x,\gamma)]\hat{\Phi}=0, \quad (34)$$

where

$$Y=(\tfrac{1}{2}\pi)^{1/2}\frac{(\omega-\omega_*-k_{\|}U'x)}{(\tau+1)\,|k_{\|}|\,v_ea_i^2}\exp\left(\frac{-\omega_*^2}{2k_{\|}^2v_e^2}\right). \quad (35)$$

In Eq. (34), the carats on $\hat{\Phi}$ and $\hat{\lambda}$ are to distinguish them from the Φ and λ of Eqs. (23) and (32) which do not include the effect of Y. Letting the prefix Δ indicate the perturbation in a quantity due to retaining Y, then $\hat{\Phi}=\Phi+\Delta\Phi$, $\gamma=\gamma_0+\delta\gamma+\Delta\gamma$, $\hat{\lambda}=[(\delta\gamma+\Delta\gamma)/a_i^2]\times[(\partial V/\partial\gamma)\,|\,\gamma_0]$; and Eqs. (23), (32), and (34) may be used to obtain

$$\Delta\gamma=-a_i^2(\tau+1)\omega_*\int_{-\infty}^{\infty}dx\ Y\Phi^2\bigg/\int_{-\infty}^{\infty}dx\ \Phi^2, \quad (36)$$

where the path of integration is chosen to provide convergence for an unstable eigenvalue. In writing (35) the contribution arising from resonant ions is neglected compared with that from the electrons because for the x of interest $\omega_*^2/k_{\|}^2v_i^2\gg1$, and this appears in the exponential of ion residue terms. Furthermore, $k_{\|}u$ is assumed to be small enough as to be negligible compared with $\omega-\omega_*$ and $k_{\|}U'x$ in Y, as the $k_{\|}u$ term is considered by Rosenbluth and Liu.[8]

The effect of the $(\tau+1)k^2a_i^2$ term of Eq. (19) on stability has been considered by Pearlstein and Berk,[9] and Liu *et al.*[10] Consequently, only the long wavelength limit in which $(\tau+1)k^2a_i^2$ is small compared with $(U'L_n)^2/4(\tau+1)v_i^2$ will be considered.

In order to evaluate the upper integral of (36) analytically, it is necessary to consider the limits in which the localization width of Φ, $|x_L|=|(2/\mu)^{1/2}|$, is either large or small compared with the shift $|x_0|$, and is large compared with x_e, where x_e is defined by $|k_{\|}(x_e)v_e/\omega_*|\equiv1$.

Using (36) to evaluate $\mathrm{Re}\Delta\gamma$ for $|x_0^2/x_L^2|\ll1$ gives, for $n=0$ and 1,

$$\mathrm{Re}\Delta\gamma=|\omega_*|\left|\left(\frac{mL_s}{ML_n}\right)^{1/2}\right|\times\begin{cases}\dfrac{U'^2L_n^2}{4(\tau+1)v_i^2}\ln\left|\left(\dfrac{2ML_n}{mL_s}\right)^{1/2}\right|+\dfrac{U'}{2\Omega_i}, & n=0\\[2ex] \dfrac{U'^2L_n^2}{4(\tau+1)v_i^2}+\dfrac{U'}{\Omega_i}, & n=1\end{cases} \quad (37)$$

provided $L_n/L_s<0$. This condition is most easily verified by using the approximate form $\Delta\gamma=-(\tau+1)\omega_*a_i^2Y(x=|x_L|)$ of Eq. (36). When $L_n/L_s>0$, the sign of the U'/Ω_i terms in $\mathrm{Re}\Delta\gamma$ is reversed and therefore the U'/Ω_i term becomes stabilizing. In obtaining the $n=0$ result,

$$\int_0^\infty\frac{dx}{x}\exp\left(-\frac{\epsilon}{x^2}+ix^2\right)=\int_0^\infty\frac{dz}{2z}\exp\left(-\frac{\epsilon}{z}+iz\right)=\int_0^{(i\epsilon)^{1/2}}\frac{dz}{z}\exp\left(-\frac{\epsilon}{z}+iz\right)$$
$$=\int_0^\infty dt\ \exp[2i(i\epsilon)^{1/2}\cosh t]=K_0[-2i(i\epsilon)^{1/2}]$$

and $\epsilon=x_e^2/|x_L^2|=|mL_s/2ML_n|\ll1$ are employed, where $\mathrm{Re}[-2i(i\epsilon)^{1/2}]>0$ is required for convergence of the integral and K_0 is a modified Bessel function of order zero.

A comparison of (33) and (37) shows that instability is not possible due to the $U'^2L_n^2/(\tau+1)v_i^2$ contribution to $\mathrm{Re}\Delta\gamma$ because $|x_0^2/x_L^2|\ll1$, where

$$\left|\frac{x_0^2}{x_L^2}\right|=\left(\frac{U'^2L_n^2}{8(\tau+1)v_i^2}\right)\left(\frac{\tau\,|L_s|}{(\tau+1)\,|L_n|}\right), \quad (38)$$

and the maximum value of $s^{-1}\ln s=e^{-1}<1$ for $s>1$. Because of $|x_0^2/x_L^2|\ll1$, however,

$$U'/\Omega_i\gg U'^2L_n^2/4(\tau+1)v_i^2$$

is to be expected so that instability due to the U'/Ω_i term in (37) is possible. Comparing the remainder of

TABLE I. Dependence of the regions of instability on the density scale length and the magnetic shear length.

Increasing magnetic shear ↓ / Increasing density gradient →	$\infty > U'\mid L_n\mid/(\tau+1)^{1/2}v_i > 1$ (Drift waves not permitted)	$1 > [U'\mid L_n\mid/(\tau+1)^{1/2}v_i] > \mid[8(\tau+1)L_n/\tau L_s]^{1/2}\mid$ (Kelvin–Helmholtz stabilized)	$\mid[8(\tau+1)L_n/\tau L_s]^{1/2}\mid > U'\mid L_n\mid/(\tau+1)^{1/2}v_i > 0$ (Kelvin–Helmholtz stabilized)
$\infty > \dfrac{\tau U'\mid L_s\mid}{(\tau+1)v_i} > \begin{cases} 2(2\tau M/\pi m)^{1/2} \\ 2\mid(ML_n/mL_s)^{1/2}(L_n/a_i)\mid \end{cases}$	Local theory: Kelvin–Helmholtz unstable $\gamma\approx\frac{1}{2}\tau^{1/2}kU'a_i,\ \tau\gg1$; $\gamma\approx\tau kU'a_i/3(\tau+1),\ \tau\lesssim1$	Drift instability driven by $\omega-\omega_*$; $\mathrm{Im}\,\omega=\mid\omega_*U'L_n/2v_i\mid(\pi m/2\tau M)^{1/2}-[(\tau+1)/\tau]\mid L_n\omega_*/L_s\mid$	Drift instability driven by $k_\parallel U'x$, Provided $L_n/L_s>0$; $\mathrm{Im}\,\omega=\mid\omega_*(mL_s/ML_n)^{1/2}\mid U'/\Omega_i-[(\tau+1)/\tau]\mid L_n\omega_*/L_s\mid$
$3 < \dfrac{\tau U'\mid L_s\mid}{(\tau+1)v_i} < \begin{cases} 2(2\tau M/\pi m)^{1/2} \\ 2\mid(ML_n/mL_s)^{1/2}(L_n/a_i)\mid \end{cases}$	Kelvin–Helmholtz unstable	Drift instability driven by $\omega-\omega_*$ stabilized by magnetic shear	Drift instability driven by $k_\parallel U'x$ stabilized by magnetic shear
$0<\tau U'\mid L_s\mid/(\tau+1)v_i<3$	Kelvin–Helmholtz stabilized	...	...

(37) and (33) for $n=0$ gives the stability condition

$$\frac{U'}{\Omega_i}<\frac{2(\tau+1)}{\tau}\left|\left(\frac{M}{m}\right)^{1/2}\left(\frac{L_n}{L_s}\right)^{3/2}\right|\approx 86\,\frac{\tau+1}{\tau}\left|\left(\frac{L_n}{L_s}\right)^{3/2}\right|, \tag{39}$$

where $(M/m)^{1/2}\approx43$ is employed. In order for (39) to be violated when $\mid x_0^2/x_L^2\mid\ll1$ is satisfied, $\mid L_n/L_s\mid<(2\tau m/M)^{1/2}(a_i/\mid L_n\mid)$ is required. In the $\mid x_0^2/x_L^2\mid\ll1$ limit, therefore, no instability due to the $\omega-\omega_*$ term in Y is possible, while the Doppler shift term $k_\parallel U'x$ can lead to instability.

In the $\mid x_0^2/x_L^2\mid\gg1$ limit, application of (36) for arbitrary mode number n yields

$$\Delta\gamma=\mid\omega_*\mid\frac{U'\mid L_n\mid}{2v_i}\left(\frac{\pi m}{2\tau M}\right)^{1/2}\left(1+\frac{\tau U'}{(\tau+1)\Omega_i}\left|\frac{L_s}{L_n}\right|\right) \tag{40}$$

provided $L_n/L_s<0$. This expression for $\Delta\gamma$ and the $L_n/L_s<0$ condition can be exactly recovered by using the seemingly approximate, but, in fact, exact, form $\Delta\gamma=-(\tau+1)\omega_*a_i^2Y(x=x_0)$ of (36). Once again when $L_n/L_s>0$, the contribution of the second term in $\Delta\gamma$ is stabilizing.

Inspection of (33) shows that $n=0$ is the least stabilized mode, and comparison of it with (40) gives the two stability conditions

$$\frac{U'}{\Omega_i}<\frac{2(\tau+1)a_i}{\mid L_s\mid}\left(\frac{2M}{\pi\tau m}\right)^{1/2}\approx 68\,\frac{(\tau+1)a_i}{\tau^{1/2}\mid L_s\mid} \tag{41}$$

and

$$\frac{U'}{\Omega_i}<\frac{\tau+1}{\tau}\left|\left(\frac{2a_i}{L_n}\right)^{1/2}\left(\frac{2\tau M}{\pi m}\right)^{1/4}\right|\approx\frac{8(\tau+1)a_i^{1/2}}{\tau^{3/4}\mid L_n^{1/2}\mid}, \tag{42}$$

where again $(M/m)^{1/2}\approx43$ is employed. Taking $\tau\approx1$, (41) and (42) show that to see these drift instabilities $a_i/\mid L_s\mid\ll1/136$ and $a_i/\mid L_n\mid\ll1/256$ are required, otherwise $U'/\Omega_i\ll1$ is violated. Because $U'\mid L_n\mid/2(\tau+1)^{1/2}v_i<1$, Eq. (42) cannot be violated unless τ is extremely large compared with one. In order for (41) to be satisfied when $\mid x_0^2/x_L^2\mid\gg1$ is also satisfied, an $\mid L_n/L_s\mid>\pi m/M$ is required, provided $U'/\Omega_i\ll1$.

DISCUSSION

If neutral injection results in U' large enough so that $U'\mid L_n\mid/(\tau+1)^{1/2}v_i>1$, then a parallel Kelvin–Helmholtz instability is excited having a growth rate $\sim kU'a_i$ (for $\tau\sim1$). Magnetic shear stabilization of this instability requires $\tau U'L_s/(\tau+1)v_i<3$, so that L_s must be less than $[3(\tau+1)^{1/2}/\tau]\mid L_n\mid$ for stabilization. This is, for example, a larger amount of magnetic shear than is presently available in tokamaks. As a result, the best means of stabilizing this parallel Kelvin–Helmholtz instability is via a strong enough density gradient so that $U'\mid L_n\mid/(\tau+1)^{1/2}v_i<1$.

As $U'\mid L_n\mid/(\tau+1)^{1/2}v_i$ decreases and becomes less than one, the parallel Kelvin–Helmholtz instability is transformed to a drift wave so that for

$$U' \mid L_n \mid /(\tau+1)^{1/2}v_i<1$$

only the much more slowly growing drift instabilities can be excited. Two drift instabilities are possible, and which one occurs depends upon whether

$$\mid x_0^2/x_L^2 \mid = [U'^2L_n^2/8(\tau+1)v_i^2][\tau \mid L_s \mid /(\tau+1) \mid L_n \mid]$$

is much less or much greater than one. For $\mid x_0^2/x_L^2 \mid \ll 1$, the Doppler shift in ω caused by U' leads to an instability if $U'/\Omega_i > 86[(\tau+1)/\tau] \mid (L_n/L_s)^{3/2} \mid$ provided $\mid L_n/L_s \mid < \tau^{1/2}a_i/30 \mid L_n \mid$, for $(M/m)^{1/2} \approx 43$. When $\mid x_0^2/x_L^2 \mid \gg 1$, a shift from the electron diamagnetic drift frequency, proportional to $(U')^2$, results in an instability which is stabilized if $U'/\Omega_i < 68[(\tau+1)/\tau^{1/2}] \times [a_i/ \mid L_s \mid]$ provided $\mid L_s/L_n \mid > 600$ and $U'/\Omega_i \ll 1$, for $(M/m)^{1/2} \approx 43$. If the magnetic shear is strong enough to stabilize the usual drift wave instability [replace $U'^2L_n^2/4(\tau+1)v_i^2$ by $(\tau+1)k^2a_i^2$ and then set $U'=0$ in (37)], then these two velocity shear driven drift wave instabilities are also stabilized. A summary of the preceding discussion is presented in Table I.

Taking the injected beam velocity equal to the critical velocity $U_{bc} \approx (m/M)^{1/3}v_e \approx (M/m)^{1/6}v_i$ discussed in the introduction, the accumulated beam injected density N_I at which the beam injected energy becomes equal to the kinetic energy of the background plasma ions is $N_I \approx (m/M)^{1/3}N$. From conservation of momentum, the mean plasma velocity along $\mathbf{B}_0$ due to the injected beam, U, is given by $U \approx N_I U_{bc}/N \approx (m/M)^{1/6}v_i$. Consequently, using $U' \equiv U/L_u$, where L_u is the scale length associated with the velocity shear, gives $U' \mid L_n \mid /v_i \approx (m/M)^{1/6} \mid L_n/L_u \mid$. As a result, for $\mid L_u/L_n \mid \ll 1$, i.e., at the beam edge, or for higher injected beam energies (more heating), Eq. (22) can be violated and the parallel Kelvin–Helmholtz instability excited.

For the parallel Kelvin–Helmholtz instability, the slab model results presented here are expected to be valid in a tokamak geometry as this instability is driven by the bulk motion of the ions and so is not expected to be significantly affected by the relatively few trapped particles. The drift wave instabilities, however, are driven by a resonance of relatively few electrons with the drift wave and because the resulting growth rates for tokamak parameters are on the order of the electron bounce frequency, the use of a slab model is only marginally justified.

ACKNOWLEDGMENTS

We wish to thank I. B. Bernstein, T. H. Stix, E. J. Valeo, and A. M. El-Nadi for important conversations; and R. B. White for an extremely valuable discussion with regard to the properties of Whittaker functions. In addition, we would like to thank J. A. Rome for bringing Ref. 6 to our attention.

Research sponsored by the Atomic Energy Commission under Grant No. AT(11-1)-3237.

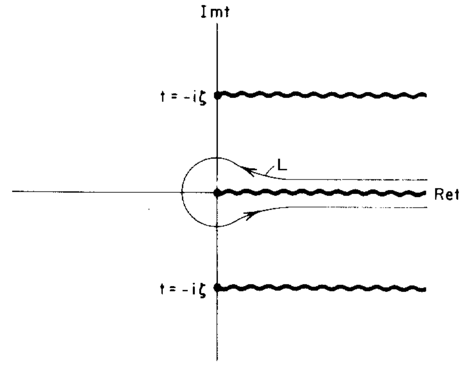

FIG. 3. The contour L in the complex t plane for $x<0$ $(\zeta<0)$. The branch cut from $t=-i\zeta$ is for the upper signs in (A6), while that from $t=+i\zeta$ is for the lower.

APPENDIX

The differential equation (14) with V given by (28) can be solved by using the WKB solution in the region about $x \approx \mid \gamma L_s/kv_i \mid$ to join the solution from the $x> \mid \gamma L_s/kv_i \mid$ limit of V, to that of the $x< \mid \gamma L_s/kv_i \mid$ limit.

In the $x> \mid \gamma L_s/kv_i \mid$ limit, the V of (14) is given by (29), and letting

$$\rho = \left(\frac{\gamma L_s W}{k a_i v_i(\tau+1)}\right)^{2/3} \left(\frac{(\tau+1)kv_i x}{W\gamma L_s} - 1\right) \quad \text{(A1)}$$

results in the Airy equation $\partial^2\Phi/\partial\rho^2 - \rho\Phi = 0$. In order for $\Phi(x \to +\infty) \to 0$, only the Airy function going to zero as $x \to +\infty$ is allowed. For $x \to -\infty$, this solution has the asymptotic form

$$\Phi(- \mid x \mid) \sim C(x)[\exp(i\tfrac{2}{3} \mid \rho \mid^{3/2} - i\tfrac{1}{4}\pi) + \exp(-i\tfrac{2}{3} \mid \rho \mid^{3/2} + i\tfrac{1}{4}\pi)], \quad \text{(A2)}$$

where the explicit x variation of $C(x)$ is not required to carry out the matching to lowest significant order.

For the $x< \mid \gamma L_s/kv_i \mid$ limit the V of (14) is given by Eq. (30). Letting

$$\zeta = (2/a_i)[x + (\tau\gamma L_s/Wkv_i)], \quad \text{(A3)}$$

results in the Whittaker equation

$$\frac{\partial^2\Phi}{\partial\zeta^2} + \left(\frac{1}{4} - \frac{\kappa}{\zeta}\right)\Phi = 0, \quad \text{(A4)}$$

where

$$\kappa = (\tau+1)\gamma L_s/2Wkv_i a_i. \quad \text{(A5)}$$

The solutions to Eq. (A4) are the Whittaker functions $W_{\pm i\kappa,1/2}(\pm i\zeta)$ which have the integral representation[11]

$$W_{\pm i\kappa,1/2}(\pm i\zeta) = -[\Gamma(\pm i\kappa)/2\pi i](\pm i\zeta)^{\pm i\kappa} \times \exp[\mp(\tfrac{1}{2}i\zeta)] \int_L dt(-t)^{\mp i\kappa} \times [1 \pm (t/i\zeta)]^{\pm i\kappa} \exp(-t), \quad \text{(A6)}$$

where the contour L is as shown in Fig. 3.

For $x \to -\infty$, $\text{Re}\zeta \to -\infty$ so that

$$[1 \pm (t/i\zeta)]^{\pm i\kappa} = 1 + (\kappa t/\zeta) + \cdots$$

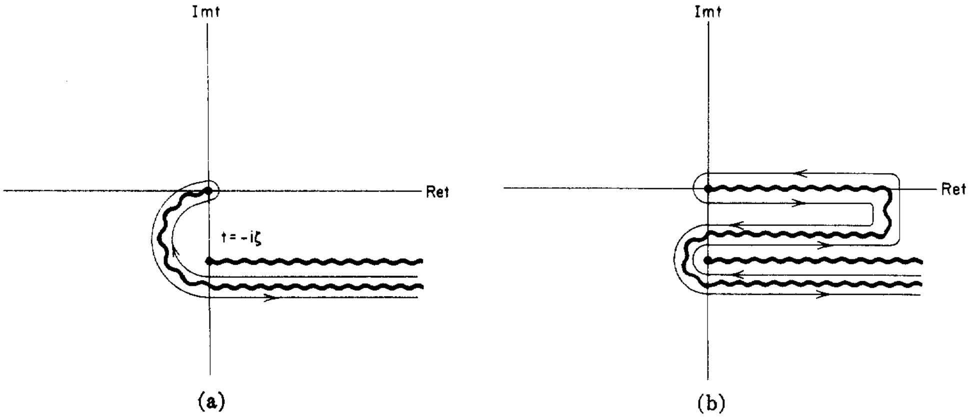

FIG. 4. The deformed branch cut from $t=0$ in the complex t plane for $x>0$ ($\zeta>0$) when $\mathrm{Im}\,\gamma>0$.

may be employed in (A6). For $\pm i\kappa$ not an integer or zero,[11]

$$[\Gamma(z)]^{-1}=(i/2\pi)\int_c dt(-t)^{-z}\exp(-t)$$

may then be used to obtain

$$W_{\pm i\kappa,1/2}(\pm i\zeta)\sim\exp[\mp\tfrac{1}{2}(i\zeta)\pm i\kappa\ln(-\zeta)+\tfrac{1}{2}(\pi\kappa)]+\cdots \tag{A7}$$

for $x\to-\infty$. In order for $\Phi(x\to-\infty)\to0$, Eq. (A7) requires $\mathrm{Re}(\pm i\kappa)<0$ which in turn gives $\pm\mathrm{Im}\gamma>0$. Consequently, for $\mathrm{Im}\gamma>0$, $W_{i\kappa,1/2}(i\zeta)$ is the appropriate solution for Φ, while for $\mathrm{Im}\gamma<0$, $W_{-i\kappa,1/2}(-i\zeta)$ is required. However, if the matching results in an eigenvalue γ having $\mathrm{Im}\gamma>0$, then because γ is the only complex quantity in (14), the complex conjugate of γ must also be an allowed eigenvalue. As a result, only $\mathrm{Im}\gamma>0$ or $\mathrm{Im}\gamma<0$ need be considered.

Taking $\mathrm{Im}\gamma>0$, the asymptotic behavior of $W_{i\kappa,1/2}(i\zeta)$ depends upon whether the branch point $t=-i\zeta$ of the integral in (A6) passes to the left or right of $t=0$ as x passes from $x<0$ to $x>0$.[12] If the branch point passes to the left of $t=0$, then the asymptotic form for $x\to+\infty$ is the same as that for $x\to-\infty$ and it is not possible to join the $x\to+\infty$ asymptotic form of the Whittaker function to (A2). Fortunately, for $\mathrm{Im}\gamma>0$, $\mathrm{Re}[-i\zeta(x=0)]>0$ so that the branch point $t=-i\zeta$ must pass to the right of $t=0$. In passing to the right of $t=0$, this branch point at $t=-i\zeta$ cannot be allowed to cross the branch cut between $t=0$ and $+\infty$. Consequently, it is convenient to deform the $t=0$ branch cut from that shown in Fig. 3 to that of Fig. 4(b) by first deforming it as shown in Fig. 4(a). Breaking the contour of Fig. 3(b) up into three sections and making the appropriate changes of variables under the integrals in (A6), $W_{i\kappa,1/2}(i\zeta)$ may then be written as

$$W_{i\kappa,1/2}(i\zeta)=-\frac{\Gamma(i\kappa)}{2\pi i}(i\zeta)^{i\kappa}\exp(-\tfrac{1}{2}i\zeta)\times\int_L dt(-t)^{-i\kappa}\left(1+\frac{t}{i\zeta}\right)^{+i\kappa}\exp(-t)$$
$$+\frac{\Gamma(i\kappa)}{2\pi i}(-i\zeta)^{-i\kappa}\exp(\tfrac{1}{2}i\zeta)[1-\exp(-2\pi\kappa)]\times\int_L dt(-t)^{+i\kappa}\left(1-\frac{t}{i\zeta}\right)^{-i\kappa}\exp(-t), \tag{A8}$$

where the contour L is the same as that shown in Fig. 3 and where $\mathrm{Im}[\zeta(x=0)]<0$ is required in order for it to be necessary to deform the contour of Fig. 3. For $x\to+\infty$, Eq. (A8) has the asymptotic form

$$W_{i\kappa,1/2}(i\zeta)\sim\exp[-\tfrac{1}{2}i\zeta+i\kappa\ln\zeta-\kappa\tfrac{1}{2}\pi]+\cdots$$
$$-[1-\exp(-2\pi\kappa)][\Gamma(i\kappa)/\Gamma(-i\kappa)]$$
$$\times\exp[\tfrac{1}{2}i\zeta-i\kappa\ln\zeta-\kappa\tfrac{1}{2}\pi]+\cdots,\qquad x\to+\infty. \tag{A9}$$

In obtaining (A9), the same procedure that precedes Eq. (A7) is employed.

To join the asymptotic forms (A2) and (A9), a WKB solution of Eq. (14) with V given by (28) is employed. To lowest order this WKB solution is

$$\Phi=A(x)\exp\left\{\frac{i}{a_i}\int_{x_w}^{x}dx'[-V(x',\gamma)]^{1/2}\right\}$$
$$+B(x)\exp\left\{-\frac{i}{a_i}\int_{x_w}^{x}dx'[-V(x',\gamma)]^{1/2}\right\}, \tag{A10}$$

where x_w is any x of the order of $|\gamma L_s/kv_i|$ and where A and B contain the higher-order corrections in x to the WKB solution, as well as arbitrary constants. Recalling $|W|\gg\tau+1$, using

$$\int_{x_w}^{x}dx'[-V]^{1/2}=\int_{x_w}^{x_2}dx[-V]^{1/2}-\int_{x}^{x_2}dx'[-V]^{1/2},$$

where $x_2=W\gamma L_s/(\tau+1)kv_i$, and taking $|x|>|x_w|$ so that the V of (29) may be used to evaluate the integral from x to x_2; the WKB solution (A10) may be joined to (A2) provided

$$A(x)\exp\left[\frac{i}{a_i}\int_{x_w}^{x_2}dx(-V)^{1/2}\right]=C(x)\exp(i\tfrac{1}{4}\pi),$$
$$B(x)\exp\left[-\frac{i}{a_i}\int_{x_w}^{x_2}dx(-V)^{1/2}\right]=C(x)\exp(-i\tfrac{1}{4}\pi). \tag{A11}$$

Equations (A11) may then be used to write the Φ of (A10) as

$$\Phi=C(x)\left[\exp\left(i\tfrac{1}{4}\pi-\frac{i}{a_i}\int_x^{x_2}dx'(-V)^{1/2}\right)+\exp\left(-i\tfrac{1}{4}\pi+\frac{i}{a_i}\int_x^{x_2}dx'(-V)^{1/2}\right)\right]. \quad (A12)$$

To join (A12) to (A9) for $|W|\gg\tau+1$,

$$\int_x^{x_2}dx'[-V]^{1/2}=\int_{x_1}^{x_2}dx[-V]^{1/2}-\int_{x_1}^{x}dx'[-V]^{1/2}$$

is employed, where $x_1=\gamma L_s/Wkv_i$. For $|x|<|x_w|$, the V of (30) may be employed in the integral from x_1 to x, and the resulting form of (A12) may be joined to (A9) provided γ satisfies the eigenvalue equation

$$-\frac{\Gamma(i\kappa)}{\Gamma(-i\kappa)}[1-\exp(-2\pi\kappa)]=\exp\left\{i\tfrac{1}{2}\pi-\frac{i2}{a_i}\times\int_{x_1}^{x_2}dx[-V(x,\gamma)]^{1/2}+i2\kappa\ln\kappa\right\}. \quad (A13)$$

In the limit in which the "barrier" width $x_1-x_p=2a_i\kappa$ is large, $|\kappa|\gg1$, Eq. (A13) reproduces the $\gamma>0$ form of the local result (20) within factors of $(\pi/2)^{1/2}$, provided $W\sim2(\tau+1)$ and $W\gg(\tau+1)$ are employed. The quantity x_p is the location of the pole of V, $x_p=-\tau\gamma L_s/Wkv_i$. The errors in the factors of $(\pi/2)^{1/2}$ are a direct result of having to neglect such terms in V in order to obtain an analytic expression for the eigenvalue equation. In addition, $|\kappa|\gg1$ results in the local result (21) being valid when $\tau U'L_s\gg8(\tau+1)v_i$.

In order to determine the effect of magnetic shear in this $|W|\gg\tau+1$ limit the eigenvalue equation (A13) may be approximately evaluated for $|\kappa|\ll1$. Because $|x_2/x_w|\approx|W/(\tau+1)|\gg1$ and $|x_1/x_w|\approx1/|W|\ll1$, $\int dx[-V]^{1/2}$ in (A13) may be evaluated to lowest significant order by simply using the V of (29) and integrating from 0 to x_2,

$$a_i^{-1}\int_{x_1}^{x_2}dx[-V]^{1/2}\approx\frac{2W\gamma L_s}{3(\tau+1)kv_ia_i}=\frac{2\tau U'L_s}{3(\tau+1)v_i}.$$

Using the preceding result, $\exp(-2\pi\kappa)\approx1-2\pi\kappa$, $\Gamma(\pm i\kappa)\approx\pm1/i\kappa$, and $\exp(i2\kappa\ln\kappa)\approx1$, Eq. (A13) reduces to

$$2\pi\kappa=\frac{\pi(\tau+1)L_s\gamma^2}{\tau k^2a_i^2v_iU'}=\exp\left(i\frac{\pi}{2}-i\frac{4\tau U'L_s}{3(\tau+1)v_i}\right). \quad (A14)$$

Writing $\gamma=|\gamma|\exp(i\psi)$, using $\exp(i\psi\pm i2\pi)=\exp(i\psi)$, and requiring stability $(\psi>\pi/2)$ for $U'=0$ gives

$$|\gamma^2|=\left(\frac{\tau ka_iU'}{\pi(\tau+1)}\right)^2\left(\frac{\pi(\tau+1)v_i}{\tau U'L_s}\right)$$

and

$$\psi=\frac{5\pi}{4}-\frac{2\tau U'L_s}{3(\tau+1)v_i}. \quad (A15)$$

For stability, $\psi>\pi/2$ is required. Consequently, the condition for shear stabilization in the $|W|\gg\tau+1$ limit is as shown in (31). In finding (31), no restrictions on τ other than $|W/\tau|\gg1$ are required. From (A14), $2\pi|\kappa|=1$, so that $\exp(-2\pi\kappa)\approx1-2\pi\kappa$ and $\exp(i2\kappa\ln\kappa)\approx1$ are not strictly valid; however, such inaccuracies only slightly change the constant on the right-hand side of Eq. (31).

Finally, note that for γ real and $>0(\psi=0)$, Eqs. (A15) give $\gamma\approx\tau kU'a_i/4(\tau+1)$ which should be compared with the local result (21). Note in particular that the $\tau\lesssim1$ restriction must be recovered for $\psi<0$ or $\tau U'L_s/(\tau+1)v_i>6$, while for $\psi\approx\pi/2$, $|W/(\tau+1)|\gg1$ is satisfied independent of τ.

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